Lesson 12. Dynamic Programming – Review

- Recall from Lessons 5-9:
  - A **dynamic program** models situations where decisions are made in a sequential process in order to optimize some objective
  - **Stages** $t = 1, 2, \dots, T$
    - stage $T \leftrightarrow$ end of decision process
  - **States** $n = 0, 1, \dots, N \leftarrow$ possible conditions of the system at each stage
  - Two representations: **shortest/longest path** and **recursive**

Shortest/longest path		Recursive
node t_n	$\leftrightarrow$	state n at stage t
edge $(t_n, (t+1)_m)$	$\leftrightarrow$	allowable decision x_t in state n at stage t that results in being in state m at stage $t+1$
length of edge $(t_n, (t+1)_m)$	$\leftrightarrow$	contribution of decision x_t in state n at stage t that results in being in state m at stage $t+1$
length of shortest/longest path from node t_n to end node	$\leftrightarrow$	value-to-go function $f_t(n)$
length of edges (T_n, end)	$\leftrightarrow$	boundary conditions $f_T(n)$
shortest or longest path	$\leftrightarrow$	recursion is min or max: $f_t(n) = \min_{x_t \text{ allowable}} \text{or} \max \left\{ \begin{pmatrix} \text{contribution of} \\ \text{decision } x_t \end{pmatrix} + f_{t+1} \begin{pmatrix} \text{new state} \\ \text{resulting} \\ \text{from } x_t \end{pmatrix} \right\}$
source node 1_n	$\leftrightarrow$	desired value-to-go function value $f_1(n)$

Example 1. Simplexville Oil needs to build capacity to refine 1,000 barrels of oil and 2,000 barrels of gasoline per day. Simplexville can build a refinery at 2 locations. The cost of building a refinery is as follows:

Oil capacity per day	Gas capacity per day	Building cost ($ millions)
0	0	0
1000	0	5
0	1000	7
1000	1000	14

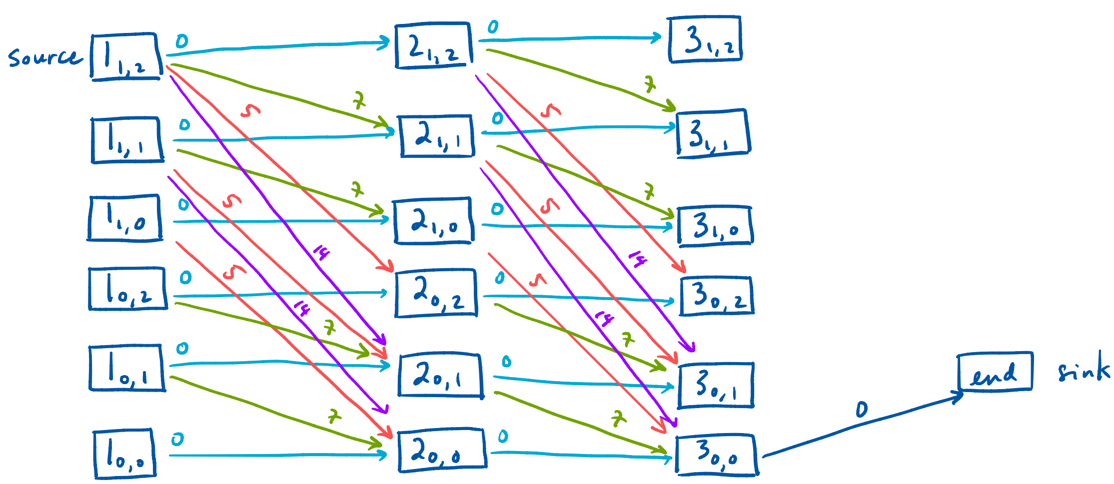
The problem is to determine how much capacity should be built at each location in order to minimize the total building cost. To make things a little simpler, assume that the capacity requirements must be met exactly.

a. Formulate this problem as a dynamic program by giving its shortest path representation.
b. Formulate this problem as a dynamic program by giving its recursive representation. Solve the dynamic program.

$$\text{Stage } t \leftrightarrow \begin{cases} \text{deciding to build at location } t & t = 1, 2 \\ \text{end of decision-making process} & t = 3 \end{cases}$$

State (n_1, n_2) $\leftrightarrow$ n_1 oil capacity and n_2 gas capacity still needed to be built $n_1 = 0, 1$; $n_2 = 0, 1, 2$

Find shortest path:



Recursive representation

- Stage t $\longleftrightarrow$ $\begin{cases} \text{deciding to build at location } t & t=1,2 \\ \text{end of decision-making process} & t=3 \end{cases}$

- State (n_1, n_2) $\longleftrightarrow$ n_1 oil capacity and n_2 gas capacity still needed to be built $\quad n_1 = 0,1$; $n_2 = 0,1,2$

- Allowable decisions x_t at stage t and state (n_1, n_2):

  $x_t = (x_{t1}, x_{t2})$ $\longleftrightarrow$ build x_{t1} oil capacity and x_{t2} gas capacity at location t

  x_t must satisfy:
  $$x_{t1} \in \{0,1\}$$
  $$x_{t2} \in \{0,1\}$$
  $$\left.\begin{array}{l} x_{t1} \le n_1 \\ x_{t2} \le n_2 \end{array}\right\} \text{ can't overbuild capacity}$$
  for $t=1,2$; $n_1 = 0,1$; $n_2 = 0,1,2$

- Contribution of x_t at stage t and state (n_1, n_2):
  $$c(x_{t1}, x_{t2}) = \begin{cases} 0 & \text{if } (x_{t1}, x_{t2}) = (0,0) \\ 5 & \text{if } (x_{t1}, x_{t2}) = (1,0) \\ 7 & \text{if } (x_{t1}, x_{t2}) = (0,1) \\ 14 & \text{if } (x_{t1}, x_{t2}) = (1,1) \end{cases}$$
  for $t=1,2$; $n_1 = 0,1$; $n_2 = 0,1,2$

- Value-to-go function

  $f_t(n_1, n_2)$ = minimum total cost to build n_1 oil capacity and n_2 gas capacity with locations $t, \ldots, 2$ available

  for $t = 1,2,3$; $n_1 = 0,1$; $n_2 = 0,1,2$

- Boundary conditions: $f_3(n_1, n_2) = \begin{cases} 0 & \text{if } (n_1, n_2) = (0,0) \\ +\infty & \text{o/w} \end{cases}$ for $n_1 = 0,1$; $n_2 = 0,1,2$.

- Recursion:
  $$f_t(n_1, n_2) = \min_{\substack{x_{t1} \in \{0,1\} \\ x_{t2} \in \{0,1\} \\ x_{t1} \le n_1 \\ x_{t2} \le n_2}} \left\{ c(x_{t1}, x_{t2}) + f_{t+1}(n_1 - x_{t1}, n_2 - x_{t2}) \right\}$$
  for $t=1,2$; $n_1 = 0,1$; $n_2 = 0,1,2$

  oil, gas

  state n_1, n_2

  decision x_{t1}, x_{t2}

  new state $n_1 - x_{t1}, n_2 - x_{t2}$

- Desired value-to-go function value: $f_1(1,2)$

$$f_t(n_1,n_2) = \min_{\substack{x_{t1}\in\{0,1\}\\ x_{t2}\in\{0,1\}\\ x_{t1}\le n_1\\ x_{t2}\le n_2}} \left\{ c(x_{t1},x_{t2}) + f_{t+1}(n_1 - x_{t1},\, n_2 - x_{t2}) \right\} \qquad c(x_{t1},x_{t2}) = \begin{cases} 0 & \text{if } (x_{t1},x_{t2}) = (0,0) \\ 5 & \text{if } (x_{t1},x_{t2}) = (1,0) \\ 7 & \text{if } (x_{t1},x_{t2}) = (0,1) \\ 14 & \text{if } (x_{t1},x_{t2}) = (1,1) \end{cases} \quad \text{for } \begin{array}{l} t=1,2 \\ n_1=0,1 \\ n_2 = 0,1,2 \end{array}$$

Solving backwards

Stage 3:
(boundary conditions)

$$f_3(n_1,n_2) = \begin{cases} 0 & \text{if } (n_1,n_2)=(0,0) \\ +\infty & \text{o/w} \end{cases} \qquad \text{for } \begin{array}{l} n_1 = 0,1 \\ n_2 = 0,1,2 \end{array}$$

Stage 2:

$$f_2(1,2) = \min_{\substack{x_{21}\in\{0,1\}\\ x_{22}\in\{0,1\}\\ x_{21}\le 1\\ x_{22}\le 2}} \left\{ c(x_{21},x_{22}) + f_3(1-x_{21},\,2-x_{22}) \right\} = \min\left\{ \begin{array}{cc} \underset{(0,0)}{\overset{0\,+\,\infty}{c(0,0)+f_3(1,2)}}, & \underset{(1,0)}{\overset{5\,+\,\infty}{c(1,0)+f_3(0,2)}} \\ \underset{(0,1)}{\overset{7\,+\,\infty}{c(0,1)+f_3(1,1)}}, & \underset{(1,1)}{\overset{14\,+\,\infty}{c(1,1)+f_3(0,1)}} \end{array} \right\} = +\infty$$

$$f_2(1,1) = \min_{\substack{x_{21}\in\{0,1\}\\ x_{22}\in\{0,1\}\\ x_{21}\le 1\\ x_{22}\le 1}} \left\{ c(x_{21},x_{22}) + f_3(1-x_{21},\,1-x_{22}) \right\} = \min\left\{ \begin{array}{cc} \underset{(0,0)}{\overset{0\,+\,\infty}{c(0,0)+f_3(1,1)}}, & \underset{(1,0)}{\overset{5\,+\,\infty}{c(1,0)+f_3(0,1)}} \\ \underset{(0,1)}{\overset{7\,+\,\infty}{c(0,1)+f_3(1,0)}}, & \underset{(1,1)}{\overset{14\,+\,0}{c(1,1)+f_3(0,0)}} \end{array} \right\} = 14$$

$$f_2(1,0) = \min_{\substack{x_{21}\in\{0,1\}\\ x_{22}\in\{0,1\}\\ x_{21}\le 1\\ x_{22}\le 0}} \left\{ c(x_{21},x_{22}) + f_3(1-x_{21},\,0-x_{22}) \right\} = \min\left\{ \underset{(0,0)}{\overset{0\,+\,\infty}{c(0,0)+f_3(1,0)}},\ \underset{(1,0)}{\overset{5\,+\,0}{c(1,0)+f_3(0,0)}} \right\} = 5$$

$$f_2(0,2) = \min_{\substack{x_{21}\in\{0,1\}\\ x_{22}\in\{0,1\}\\ x_{21}\le 0\\ x_{22}\le 2}} \left\{ c(x_{21},x_{22}) + f_3(0-x_{21},\,2-x_{22}) \right\} = \min\left\{ \underset{(0,0)}{\overset{0\,+\,\infty}{c(0,0)+f_3(0,2)}},\ \underset{(0,1)}{\overset{7\,+\,\infty}{c(0,1)+f_3(0,1)}} \right\} = +\infty$$

$$f_2(0,1) = \min_{\substack{x_{21}\in\{0,1\}\\ x_{22}\in\{0,1\}\\ x_{21}\le 0\\ x_{22}\le 1}} \left\{ c(x_{21},x_{22}) + f_3(0-x_{21},\,1-x_{22}) \right\} = \min\left\{ \underset{(0,0)}{\overset{0\,+\,\infty}{c(0,0)+f_3(0,1)}},\ \underset{(0,1)}{\overset{7\,+\,0}{c(0,1)+f_3(0,0)}} \right\} = 7$$

$$f_2(0,0) = \min_{\substack{x_{21}\in\{0,1\}\\ x_{22}\in\{0,1\}\\ x_{21}\le 0\\ x_{22}\le 0}} \left\{ c(x_{21},x_{22}) + f_3(0-x_{21},\,0-x_{22}) \right\} = \min\left\{ \underset{(0,0)}{\overset{0\,+\,0}{c(0,0)+f_3(0,0)}} \right\} = 0$$

Stage 1:

$$f_1(1,2) = \min_{\substack{x_{11}\in\{0,1\}\\ x_{12}\in\{0,1\}\\ x_{11}\le 1\\ x_{12}\le 2}} \left\{ c(x_{11},x_{12}) + f_2(1-x_{11},\,2-x_{12}) \right\} = \min\left\{ \begin{array}{cc} \underset{(0,0)}{\overset{0\,+\,\infty}{c(0,0)+f_2(1,2)}}, & \underset{(1,0)}{\overset{5\,+\,\infty}{c(1,0)+f_2(0,2)}} \\ \underset{(0,1)}{\overset{7\,+\,14}{c(0,1)+f_2(1,1)}}, & \underset{(1,1)}{\overset{14\,+\,7}{c(1,1)+f_2(0,1)}} \end{array} \right\} = 21$$

Optimal value: $f_1(1,2) = 21$ $\Rightarrow$ Minimum total cost of building 1000 oil capacity + 2000 gas capacity = \$21 million

Optimal solution: $(x_{11}, x_{12}) = (1,1)$ $\Rightarrow$ At location 1, build 1000 oil capacity
1000 gas capacity

$(x_{21}, x_{22}) = (0,1)$ $\Rightarrow$ At location 2, build 1000 gas capacity